PMM U.S.S.R.,Vol.44,pp.733-738 0021-8928/81/6 0733 $7.50/0

UDC 532.516:536.25

ON CONVECTION IN A FLUID FILLING THE CAVITY OF A MOVING SOLID BODY*

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The problem of simultaneous motion about a fixed point O of a solid body and unevenly heated viscous incompressible fluid completely filling a finite cavity of the body is considered in linear formulation. The center of mass of the system body plus fluid in the state of mechanical equilibrium is assumed to coincide with point O. The theorem on solvability of the Cauchy problem for small unsteady equilibrium perturbations is proved, and normal perturbations and the spectrum of the problem arising in the analysis of such perturbations are investigated. It is shown that the whole spectrum consists of normal eigenvalues and lies in some half-band containing the real axis. It is shown that the respective system of root vectors is complete. Properties of the spectrum and the dependence on Rayleigh numbers are investigated. Rayleigh numbers for which the real parts of eigenvalues are positive, i.e. when the generated oscillating normal perturbations are damped in time, are evaluated in the case of the fluid being heated from below and above.

1. Statement of the problem. Let a solid body with a cavity completely filled with nonuniformly heated viscous incompressible fluid move about a fixed point O. The system body plus fluid is heated up so as to obtain mechanical equilibrium and bring the center of mass to the fixed point O.

We introduce the fixed orthogonal coordinate system $Oy_1y_2y_3$ (with the y_3-axis directed upward) and a moving coordinate system $Ox_1x_2x_3$ rigidly attached to the body.

In the coordinate system $Ox_1x_2x_3$ the equations of heat convection that define fluid motion in the Boussinesq approximation are of the form

$$\mathbf{u}' + (\mathbf{u}, \nabla)\ \mathbf{u} + \boldsymbol{\omega} \times (\mathbf{w} \times \mathbf{r}) + \boldsymbol{\varepsilon} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{u} = -\rho_0^{-1}\nabla p + \nu\Delta\mathbf{u} + g\beta\mathbf{k}_3 T,\ T' + (\mathbf{u}, \nabla T) = \chi\Delta T,\ \operatorname{div}\mathbf{u} = 0, \quad (x \in \Omega) \tag{1.1}$$

where $\mathbf{u}$ is the fluid relative velocity vector, $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ are the angular velocity and acceleration of the body, T is the temperature read from its mean constant value T_1, p represents pressure deviation from the hydrostatic pressure p_1 which corresponds to the constant temperature T_1, $\rho = \rho_0(1 - \beta T)$ is the fluid density, ν, β, χ are the coefficients of kinematic viscosity, thermal expansion, and thermal conductivity, respectively, $\mathbf{k}_3$ is the unit vector of axis Oy_3, $\mathbf{r}$ is the radius vector relative to point O, and Ω is the bounded region filled with fluid.

Let us clarify the conditions under which mechanical equilibrium is possible, i.e. the body with fluid is stationary. We set in Eqs. (1.1) the relative and angular velocities equal zero and shall seek the steady temperature and pressure distribution in the state of mechanical equilibrium. Denoting by T_0 and p_0 the equilibrium distribution of temperature and pressure distribution, from (1.1) we then obtain

$$-\rho_0^{-1}\nabla p_0 + g\beta\,\mathbf{k}_3 T = 0, \qquad \nabla T_0 = 0$$

As shown in /1/, temperature T_0 changes linearly with height

$$T_0 = -ay_3 + b \tag{1.2}$$

where a and b are constants.

We linearize system (1.1) about the equilibrium position

$$\mathbf{u}_0 = 0,\ \boldsymbol{\omega}_0 = 0,\ T_0 = -ay_3 + b,\ -\rho_0^{-1}\nabla p_0 + g\beta\mathbf{k}_3 T_0 = 0$$

and obtain

$$\mathbf{u}' + P(\boldsymbol{\varepsilon} \times \mathbf{r}) = -\nabla p + \Delta\mathbf{u} + RT\mathbf{k}_3,\ \operatorname{div}\mathbf{u} = 0, \quad T' \mp P^{-1}\Delta T + P^{-1}(\mathbf{k}_3, \mathbf{u}),\ P = \frac{\nu}{\chi},\ R = \frac{g\beta a L^4}{\nu\chi} \tag{1.3}$$

which is in dimensionless form, with P denoting the Prandtl number, R the Rayleigh number, and L representing a characteristic linear dimension of region Ω.

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*Prikl.Matem.Mekhan.,44,No.6,1027-1034,1980

The thermal conductivity of the vessel wall is considered to be considerably higher than that of the fluid, it is therefore possible to assume an unchanging equilibrium distribution of temperature with its perturbation vanishing along the cavity wall. At the cavity wall the following conditions

$$\mathbf{u} = 0, \quad T = 0 \quad \text{on} \quad S \tag{1.4}$$

are then satisfied.

We denote by M_0 and M_1 the mass of the body and fluid, respectively, and by $\mathbf{r}_0$, $\mathbf{r}_1$ the radius vectors of their centers of inertia in the unperturbed state relative to point O. With an accuracy to terms of second order of smallness we have

$$M_0\mathbf{r}_0 + M_1\mathbf{r}_1 = M_0\mathbf{r}_0 + \rho_0 \int_\Omega \mathbf{r}(1-\beta T_0 - \beta T)\, d\Omega = M_2\mathbf{r}_2 - \rho_0\beta \int_\Omega \mathbf{r} T\, d\Omega = -\rho_0\beta \int_\Omega \mathbf{r} T\, d\Omega \tag{1.5}$$

where $M_2 = M_0 + M_1$ is the mass of the complete system and $\mathbf{r}_2$ is the radius vector of the system body plus fluid in the unperturbed state, which by definition is zero.

In the case considered here the system body plus fluid is subjected only to the gravity force moment which is induced by the displacement of the center of inertia owing to the non-uniform heating of the fluid in the perturbed motion. Then, by analogy with /2,3/ and with allowance for (1.5), the linearized equation of motion of the body with heated fluid written in dimensionless form is

$$J\boldsymbol{\varepsilon} + PG\frac{d}{dt}\int_\Omega \mathbf{r} \times \mathbf{u}\, d\Omega - RP^{-1}G\Big(\mathbf{k}_3 \times \int_\Omega \mathbf{r} T\, d\Omega\Big) = 0, \quad G = \frac{\rho_0}{\rho_1} \tag{1.6}$$

where G is a dimensionless quantity, ρ_1 is the mean density of the body plus fluid, and J is the dimensionless moment of inertia in the mechanical equilibrium state.

Let us investigate the problem of determination of the motion of the body with heated fluid (1.3), (1.4), and (1.6) with the initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad T|_{t=0} = T^\circ, \quad \boldsymbol{\omega}|_{t=0} = \boldsymbol{\omega}_0 \tag{1.7}$$

2. The theorem of existence. We denote by $L_{2,0}(\Omega)$ the closure in the L_2-norm of the set of all smooth solenoidal vector functions $\mathbf{v}$ that satisfy on S the condition $\mathbf{v}_n = 0$. It was shown in /4/ that the orthogonal complement $L_{2,0}(\Omega)$ in L_2 is the closure in the L_2-norm of gradients of all smooth functions in Ω.

We introduce the space $W_{2,0}^1(\Omega)$ which is obtained as the supplement of the set of infinitely differentiable finite in Ω solenoidal vectors in the metric that corresponds to the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\, d\Omega$$

We denote by $H_2(\Omega)$ the Hilbert space consisting of all functions summable in quadrature over region Ω, and by $H_2^1(\Omega)$ the Sobolev space with the norm

$$\|T\|^2 = \int_\Omega |\operatorname{grad} T|^2 d\Omega + \int_S |T|^2 dS$$

Let $H_{2,0}^1(\Omega)$ be a subspace of $H_2^1(\Omega)$ of functions that vanish on S.

Let Π be the orthogonal projector from $L_2(\Omega)$ into $L_{2,0}(\Omega)$. It was shown in /5,6/ that the operator $-\Pi\Delta$ in $W_{2,0}^1(\Omega)$ can be extended to the self-conjugate positive definite operator A, and in /1/ that the operator Δ can be extended according to Friedrichs to the self-conjugate positive definite operator G.

Let us transform the system of Eqs. (1.3), (1.6). For this we determine using (1.6) the angular acceleration $\boldsymbol{\varepsilon}$ which we substitute into the first equation of system (1.3). Then act on the obtained equation by the operator Π. This yields

$$(I + B)\mathbf{u}' = -A\mathbf{u} + R(S_1 + B_1)T, \qquad T' = -P^{-1}GT + P^{-1}S_2\mathbf{u}, \quad B\mathbf{v} = G\Pi\Big(\mathbf{r} \times J^{-1}\int_\Omega \mathbf{r} \times \mathbf{v}\, d\Omega\Big) \tag{2.1}$$

$$S_1 T = \Pi\mathbf{k}_3 T, \qquad B_1 v = -G\Pi\,(r \times J^{-1}(k_3 \times \int_\Omega rT d\Omega)), \quad S_2 v = (k_3, v)$$

It is convenient to consider system (2.1) as a single ordinary differential equation in the Hilbert space $L_{2,0}(\Omega) \times H_2(\Omega)$, namely the equation

$$Q\eta'(t) + M\eta(t) + N\eta(t) = 0 \tag{2.2}$$

where operators Q, M, N are defined by matrices

$$Q=\begin{Vmatrix} I+B & 0 \\ 0 & I \end{Vmatrix}, \quad M=\begin{Vmatrix} A & 0 \\ 0 & P^{-1}C \end{Vmatrix}, \quad N=\begin{Vmatrix} 0 & -R(S_1+B_1) \\ -P^{-1}S_2 & 0 \end{Vmatrix}, \quad \eta(t)=\begin{Vmatrix} u \\ T \end{Vmatrix}$$

We supplement Eq. (2.2) by the initial condition

$$\eta(0)=\eta_0=\operatorname{col}(u_0,\ T^0) \tag{2.3}$$

We denote by $B_2=B_2([0,\ T],\ L_{2,0}\times H_2)$ the space of all highly measurable in $L_{2,0}\times H_2$ functions $\eta(t)=\operatorname{col}(u,\ T)$ for which the norm

$$\|\eta(t)\|_{B_2}=\Big(\int_0^T(\|u\|^2_{L_2}+\|T\|^2_{H_2})\,dt\Big)^{1/2}$$

is finite.

It was shown in /3,8/ that operator B is self-conjugate and negative in $L_{2,0}(\Omega)$, and $\|B\|<1$. It is evident that

$$((I+B)\ u,\ u)\geqslant\|u\|^2-|(Bu,\ u)|\geqslant(1-\|B\|)\|u\|^2$$

Then Q is the self-conjugate positive definite operator

$$(Q\eta,\ \eta)\geqslant(1-\|B\|)\|\eta\|^2 \tag{2.4}$$

Since operator B is bounded in $L_{2,0}(\Omega)$ hence

$$\|Q\eta\|\leqslant(1+\|B\|)\|\eta\| \tag{2.5}$$

On the basis of the above reasoning it is possible to consider in the space $L_{2,0}\times H_2$ the operators $Q^{1/2}$ and $Q^{-1/2}$. Carrying out in Eq. (2.2) the substitution $\eta(t)=Q^{-1/2}\xi$ we reduce problem (2.2), (2.3) to the form

$$\xi'+\Phi\xi+F\xi=0,\ \ \xi(0)=\xi_0=Q^{1/2}\eta_0, \qquad \Phi=Q^{-1/2}MQ^{-1/2},\ \ F=Q^{-1/2}NQ^{-1/2} \tag{2.6}$$

As the solution of problem (2.6) in B_2 we understand the absolutely continuous function $\xi(t)$ which at almost all t satisfies the equation and initial condition (2.5), and such $\xi'(t)$, $\Phi\xi+F\xi$ belong to B_2.

Theorem 1. Let ξ_0 belong to the determination region of operator Φ then problem (2.6) has a unique solution in B_2.

Proof. Let $\zeta(t)$ be an arbitrary function in B_2, then from (2.5) we obtain

$$\|\zeta(t)\|\leqslant(1+\|B\|)\|Q^{-1}\zeta(t)\|$$

By setting in it $\zeta=Q^{1/2}\mu$ and taking into account (2.4) we obtain

$$\|Q^{-1/2}\mu\|(1+\|B\|)\geqslant\|Q^{1/2}\mu\|=(Q^{1/2}\mu,\ Q^{1/2}\mu)^{1/2}=(Q\mu,\ \mu)^{1/2}>(1-\|B\|)^{1/2}\|\mu\| \tag{2.7}$$

Operator Φ is by definition self-conjugate in $L_{2,0}\times H_2$. We shall prove its positive definiteness using (2.7) and the positive definiteness of operator M. We have

$$(\Phi\ \mu,\ \mu)=(MQ^{-1/2}\mu,\ Q^{-1/2}\mu)\geqslant\gamma\|Q^{-1/2}\mu\|^2\geqslant\gamma(1-\|B\|)(1+\|B\|)^{-2}\|\mu\|^2$$

Operator F is completely subordinated to operator Φ /9/. Indeed, operator F can be represented as $F=Q^{-1/2}NM^{-1}Q^{1/2}\Phi$, where operator $Q^{-1/2}NM^{-1}Q^{1/2}$ is entirely continuous in $L_{2,0}(\Omega)\times H_2(\Omega)$ as the product of the entirely continuous operator M^{-1} and bounded operators.

It follows from the results in /9/ that the semigroup generated by operator $\Phi+F$ is analytic. The proof of the theorem now follows from /10/.

If $\xi(t)$ is a solution of problem (2.6), we consider function $\eta(t)=Q^{-1/2}\xi(t)$ as the generalized solution of problem (1.3), (1.4), (1.6), (1.7).

3. Normal perturbations. Let us consider the normal perturbations of the nonuniformly heated fluid associated with the motion of the system body plus fluid, i.e. we shall study particular solutions of the problem which depend on time in conformity with an exponential law

$$(u,\ p,\ T)=\exp(-\lambda t)(u_1,\ p_1,\ T_1)$$

where $u_1,\ p_1,\ T_1$ are functions of coordinates only. For these functions we obtain from (2.1) the system of equations

$$\begin{aligned}-\lambda(I+B)\ u_1&=-Au_1+R(S_1+B_1)\ T_1\\ -\lambda\ T_1&=-P^{-1}C\ T_1+P^{-1}S_2\ u_1\end{aligned} \tag{3.1}$$

which can be represented in the form of a single equation in a Cartesian product of Hilbert spaces $L_{2,0}(\Omega)$ and $H_2(\Omega)$, namely

$$-\lambda Q\eta + M\eta + N\eta = 0,\ \eta = \mathrm{col}\,(u_1,\ T_1)$$

Setting $\eta = Q^{-1/2}\xi$ we obtain

$$-\lambda\xi + \Phi\xi + F\xi = 0 \tag{3.2}$$

It is clear from the construction that the spectra of Eqs.(3.1) and (3.2) are the same. Let us investigate the spectrum disposition in the plane and its dependence on the Rayleigh number.

Theorem 2. The entire spectrum of problem (3.1) consists of normal eigenvalues $\lambda_1, \lambda_2, \ldots$ and lies in the region

$$\mathrm{Re}\,\lambda \geqslant \lambda_{\max}(Q)\,\lambda_{\min}(M) - \|N\|\,\lambda_{\min}(Q), \quad |\mathrm{Im}\,\lambda| \leqslant \|N\|\,\lambda_{\min}(Q) \tag{3.3}$$

There exists not more than a finite number of eigenvalues

λ_k with $\mathrm{Re}\,\lambda_k < 0$, as $k \to \infty$, $\mathrm{Re}\,\lambda_k \to +\infty$

The system of root vectors of problem (3.1) is complete in the space $W_{2,0}^1(\Omega) \times H_{2,0}^1(\Omega)$.

Proof. Operator Φ is self-conjugate with a discrete spectrum /11/, so that $\Phi^{-1} = Q^{1/2}M^{-1}Q^{1/2}$ is a self-conjugate completely continuous operator in $L_{2,0}\times H_2$. Then $F\Phi^{-1} = Q^{-1/2}NM^{-1}Q^{1/2}$ is a completely continuous operator, i.e. F is the completely continuous operator Φ /11/.

Let us consider the identity

$$\Phi^{-1}F\Phi^{-1} = (Q^{1/2}M^{-1}Q^{1/2})(Q^{-1/2}NQ^{-1/2})(Q^{1/2}M^{-1}Q^{1/2}) = Q^{1/2}M^{-1}NM^{-1}Q^{1/2}$$

Then by virtue of properties of the s-numers we have

$$s_n(\Phi^{-1}F\Phi^{-1}) = s_n(Q^{1/2}M^{-1}NM^{-1}Q^{1/2}) \leqslant \|Q^{1/2}M^{-1}N\|\,\|Q^{1/2}\| \times s_n(M^{-1}) \tag{3.4}$$

Since M^{-1} acts from $L_{2,0}\times H_2$ into $W_{2,0}^1 \times H_{2,0}^1$, hence it follows in conformity with /12/ that

$$c_1 n^{-1/2} \leqslant s_n(M^{-1}) \leqslant c_2 n^{-1/2} \tag{3.5}$$

Inequalities (3.4) and (3.5) imply that operator $\Phi^{-1}F\Phi^{-1}$ is of finite order. Then from theorem 10.1 in /11/ follows that the spectrum consists of normal eigenvalues and that the system of root vectors of problem (3.1) is complete in the space $W_{2,0}^1(\Omega) \times H_{2,0}^1(\Omega)$.

Representing the bounded operator F in the form

$$F = \mathrm{Re}\,F + i\,\mathrm{Im}\,F, \quad \mathrm{Re}\,F = \frac{F+F^*}{2}, \quad \mathrm{Im}\,F = \frac{F-F^*}{2i}$$

we obtain from (3.2)

$$(\mathrm{Re}\,\lambda + i\,\mathrm{Im}\,\lambda)\,\|\xi\|^2 = (\Phi\xi, \xi) + (\mathrm{Re}\,F\xi, \xi) + i\,(\mathrm{Im}\,F\xi, \xi)$$

From which

$$\mathrm{Re}\,\lambda\,\|\xi\|^2 = (\Phi\xi, \xi) + (\mathrm{Re}\,F\xi, \xi), \quad \mathrm{Im}\,\lambda\,\|\xi\|^2 = (\mathrm{Im}\,F\xi, \xi)$$

and then

$$\mathrm{Re}\,\lambda = \frac{(\Phi\xi, \xi)}{\|\xi\|^2} + \frac{(\mathrm{Re}\,F\xi, \xi)}{\|\xi\|^2} \geqslant \frac{(MQ^{-1/2}\xi, Q^{-1/2}\xi)}{\|\xi\|^2} - \frac{|(\mathrm{Re}\,F\xi, \xi)|}{\|\xi\|^2} \tag{3.6}$$

Moreover, if we set $\mu = Q^{-1/2}\xi$, then

$$\frac{(MQ^{-1/2}\xi, Q^{-1/2}\xi)}{\|\xi\|^2} = \frac{(M\mu, \mu)}{\|Q^{1/2}\mu\|^2} \geqslant \frac{\|\mu\|^2}{\|Q^{1/2}\mu\|^2}\inf_{\mu}\frac{(M\mu, \mu)}{\|\mu\|^2} = \tag{3.7}$$

$$\lambda_{\min}(M)\frac{\|\mu\|^2}{\|Q^{1/2}\mu\|^2} \geqslant \lambda_{\min}(M)\sup_{\mu}\frac{\|Q^{1/2}\mu\|^2}{\|\mu\|^2} = \lambda_{\min}(M)\,\lambda_{\max}(Q)$$

Let us evaluate the second term by analogy with (3.7)

$$\frac{|(\mathrm{Re}\,F\xi, \xi)|}{\|\xi\|^2} \leqslant \frac{|(F\xi, \xi)|}{\|\xi\|^2} = \frac{|(NQ^{-1/2}\xi, Q^{-1/2}\xi)|}{\|\xi\|^2} \leqslant \|N\|\,\lambda_{\min}(Q) \tag{3.8}$$

Then using inequalities (3.6) – (3.8) we obtain the first of inequalities (3.5). The second of inequalities (3.5) is similarly proved. The theorem is proved.

Theorem 3. Let the fluid be heated from beneath, i.e. the Rayleigh number R is positive. If it satisfies the inequality

$$R \leqslant 4\lambda_{\min}(A)\lambda_{\min}(C)(2+GJ_0J_{33}^{-1})^{-2}, \quad J_0 = \int_\Omega \mathbf{r}^2 d\Omega \tag{3.9}$$

where J_0 is the dimensionless polar moment of inertia of the fluid relative to point O and J_{33} is the smallest component of the moment of inertia J, then the complete spectrum of problem (3.1) lies in the region

$$\operatorname{Re}\lambda > 0, \quad |\operatorname{Im}\lambda| \leqslant \|N\|\lambda_{\min}(Q)$$

Proof. We transform system (3.1) to the form

$$-\lambda(I+B)\mathbf{u} + A\mathbf{u} = R(S_1+B_1)T, \qquad -\lambda PRT + RCT = RS_2\mathbf{u}$$

If λ is the eigenvalue of system (3.1) to which corresponds the eigenfunction $\eta = \operatorname{col}(\mathbf{u}, T)$, then the equalities

$$-\lambda\|(I+B)^{1/2}\mathbf{u}\|^2 + \|A^{1/2}\mathbf{u}\|^2 = R[(S_1, T, \mathbf{u}) + (B_1T, \mathbf{u})], \quad -\lambda PR\|T\|^2 + R\|C^{1/2}T\|^2 = R(S_2\mathbf{u}, T) \tag{3.10}$$

are valid. It follows from this that

$$\operatorname{Re}\lambda(\|(I+B)^{1/2}\mathbf{u}\|^2 + PR\|T\|^2) = \|A^{1/2}\mathbf{u}\|^2 + R\|C^{1/2}T\|^2 - R\operatorname{Re}[(S_1T, \mathbf{u}) + (B_1, T, \mathbf{u}) + (S_2, \mathbf{u}, T)] \tag{3.11}$$

By virtue of the construction of operators S_1, S_2, B_1

$$\begin{aligned} &|\operatorname{Re}(S_1T, \mathbf{u})| \leqslant |(S_1T, \mathbf{u})| \leqslant \|T\|\|\mathbf{u}\| \\ &|\operatorname{Re}(S_2\mathbf{u}, T)| \leqslant |(S_2\mathbf{u}, T)| \leqslant \|T\|\|\mathbf{u}\| \\ &|\operatorname{Re}(B_1T, \mathbf{u})| \leqslant |(B_1T, \mathbf{u})| \leqslant GJ_0J_{33}^{-1}\|T\|\|\mathbf{u}\| \end{aligned} \tag{3.12}$$

Then from (3.12) and (3.11) follows that

$$\operatorname{Re}\lambda(\|(I+B)^{1/2}\mathbf{u}\|^2 + PR\|T\|^2) \geqslant \|A^{1/2}\mathbf{u}\|^2 + R\|C^{1/2}T\|^2 - R(2+GJ_0J_{33}^{-1})\|T\|\mathbf{u}\| \tag{3.13}$$

Since operators A and C are positive definite and self-conjugate, hence

$$\|A^{1/2}\mathbf{u}\|^2 = (A\mathbf{u}, \mathbf{u}) \geqslant \lambda_{\min}(A)\|u\|^2, \ \|C^{1/2}T\|^2 \geqslant \lambda_{\min}(C)\|T\|^2$$

From this and (3.13) we obtain the inequality

$$\operatorname{Re}\lambda(\|(I+B)^{1/2}\mathbf{u}\|^2 + PR\|T\|^2) \geqslant \lambda_{\min}(A)\|\mathbf{u}\|^2 + R\lambda_{\min}(C)\|T\|^2 - R(2+GJ_0J_{33}^{-1})\left(\frac{1}{2}\varepsilon_1\|\mathbf{u}\|^2 + \frac{1}{2\varepsilon_1}\|T\|^2\right)$$

Setting $\varepsilon_1 = 2\lambda_{\min}(A)(2+GJ_0J_{33}^{-1})^{-1}R^{-1}$ we obtain the following valid estimate for $\operatorname{Re}\lambda$:

$$\operatorname{Re}\lambda(\|(I+B)^{1/2}\mathbf{u}\|^2 + PR\|T\|^2) \geqslant \left(R\lambda_{\min}(C) - \frac{R^2(2+GJ_0J_{33}^{-1})^2}{4\lambda_{\min}(A)}\right)\|T\|^2$$

This and the conditions of the theorem constitute the proof.

Thus, when condition (3.9) is satisfied and the fluid is heated from below, the oscillating normal perturbations are dampened. In this case

$$\operatorname{Im}\lambda = -R\operatorname{Im}(B_1T, \mathbf{u})(\|(I+B)^{1/2}\mathbf{u}\|^2 + PR\|T\|^2)^{-1}$$

The last formula shows that the oscillating perturbations are induced by operator B_1 associated with the transport force. When the body heated from underneath is stationary, there are no normal oscillating perturbations.

If there is no temperature gradient, $R = 0$, then it follows from (3.1) that the entire spectrum consists of real positive numbers with a single limit point $+\infty$. In that case all perturbations are monotonically damped, i.e. equilibrium of the system body plus fluid is stable.

Theorem 4. Let the fluid be heated from above, i.e. the Rayleigh number R is negative. If it satisfies the inequality

$$|R| < 4\lambda_{\min}(A)\lambda_{\min}(C)J_{33}^2G^{-2}J_0^{-2} \tag{3.14}$$

the entire spectrum of problem (3.1) lies in the region

$$\operatorname{Re}\lambda > 0, \quad |\operatorname{Im}\lambda| \leqslant \|N\|\lambda_{\min}(Q)$$

Proof. From (3.10) we have the equality

$$-\lambda(\|(I+B)^{1/2}u\|^2 - PR\|T\|^2) + \|A^{1/2}\mathbf{u}\|^2 - R\|C^{1/2}T\|^2 = R[(S_1T, \mathbf{u}) + (B_1T, \mathbf{u}) - (S_2\mathbf{u}, T)]$$

It is obvious that

$$(S_1, T, \mathbf{u}) = (\overline{S_2\mathbf{u}, T})$$

From this and the preceding equality we have

$$\operatorname{Re}\lambda = \frac{\|A^{1/2}\mathbf{u}\|^2 - R\,\|C^{1/2}T\|^2 - R\,\operatorname{Im}(B_1T, \mathbf{u})}{\|(I+B)^{1/2}\mathbf{u}\|^2 - PR\,\|T\|^2} \tag{3.15}$$

As in the proof of Theorem 3, we obtain

$$\|A^{1/2}\mathbf{u}\| - R\,\|C^{1/2}T\|^2 - R\,\operatorname{Im}(B_1T, \mathbf{u}) \geqslant |R|\,(\lambda_{\min}(C) - |R|\,G^2J_0{}^24^{-1}\lambda_{\min}^{-1}(A)\,J_{33}^{-2})\,\|T\|^2 \tag{3.16}$$

Proof of the theorem follows from inequalities (3.14) and (3.16).

When the body is stationary, operator $B_1 = 0$, then (3.15) implies that $\operatorname{Re}\lambda > 0$ and motion of the fluid is stable. This fact is known from the theory of free convection (see e.g. /1/).

The authors thank S. G. Krein and N. G. Kopachevskii for discussing this paper, and the reviewer for his remarks.

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Translated by J.J.D.